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LETTER TO THE EDITOR

Self-fractional Fourier functions and selection of modes

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Abstract. It is shown that any function from L^2 can be represented as a linear combination of M self-fractional Fourier functions of order M, which are orthogonal to each other. Each of them contains a selection of Hermite–Gauss modes of the generator function. The physical meaning of the synthesis of self-fractional Fourier functions is discussed.

1. Introduction

Self-fractional Fourier functions (SFFFs) [1–4], which are invariant under the fractional Fourier transform (fractional FT) [5] for some angle, are closely related to imaging phenomena in first-order optical systems and has applications in the theory of laser resonator modes. They are eigenfunctions of the corresponding fractional FT operator. SFFFs cover, as a particular case, self-Fourier functions (SFFs) [6–9], whose Fourier transforms (FTs) are identical to themselves. It has been shown [1, 2] how to generate a SFFF from any transformable function for some angle $2\pi N/M$, where M and N are relatively prime integers. Moreover, a SFFF for an angle $2\pi N/M$ is also one for angles $2\pi j/M$, $j = 1, 2, \ldots$ [2]. This allows us to define a SFFF of order M. The only SFFFs for angles α such that $\alpha/2\pi$ is an irrational number are the Hermite–Gauss functions [3].

Here we will prove that any function from L^2 can be represented as a sum of M self-fractional Fourier functions of order M which are orthogonal to each other. Also it will be shown that M orthogonal SFFFs of order M can be constructed from any fractional Fourier transformable generator function. We will show that the procedure for the construction of a SFFF corresponds to a mode filtration of the related generator function, which in the limit case $M \to \infty$ reduces to a one mode selection.

2. Some definitions

The fractional FT at an angle α of a function $f(x)$ is given as follows [10, 11]

$$[R^\alpha f(x)](u) = \int_{-\infty}^{\infty} f(x) K_\alpha(x, u)\, \mathrm{d}x \tag{1}$$

with the kernel

$$K_\alpha(x, u) = \sqrt{\frac{1 - \mathrm{i}\cot\alpha}{2\pi}} \exp\left(\mathrm{i}\frac{\cos\alpha(x^2 + u^2) - 2xu}{2\sin\alpha}\right). \tag{2}$$

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This transform describes, except for a phase shift $\alpha/2$, the time evolution of the wave function of the harmonic oscillator [12], as well as the wave propagation through a quadratic refractive index medium in paraxial approximation [11–14]. If α or $\alpha+\pi$ is a multiple of 2π, the kernel $K_\alpha(x,u)$ reduces to $\delta(x-u)$ or $\delta(x+u)$, respectively. Thus the fractional FT at angle $2\pi n$ (n is an integer) corresponds to the identity operator. For $\alpha = \pi/2$, relationship (1) is the ordinary Fourier transform (FT).

A function $f_\alpha(x)$ is a self-fractional Fourier function for angle α if it satisfies the following equation

$$R^\alpha[f_\alpha(x)](u) = Af_\alpha(u) \tag{3}$$

where A is a complex constant factor. In other words, $f_\alpha(x)$ is an eigenfunction of the corresponding fractional FT operator R^α with eigenvalue A. From (1), (2) and (3) it follows that any arbitrary function is a SFFF for $\alpha = 2\pi n$, a symmetric (even or odd) function is SFFF for $\alpha = \pi n$, and that a SFF is SFFF for $\alpha = \pi n/2$, where n is an integer.

It is well known from quantum mechanics (see, e.g., [12]) that the Hermite–Gauss functions $\Psi_n(u)$

$$\Psi_n(u) = (\sqrt{\pi}2^n n!)^{-1/2}\exp(-u^2/2)H_n(u) \tag{4}$$

where $H_n(u)$ are the Hermite polynomials, are SFFFs for any angle α

$$R^\alpha[\Psi_n(u)](x) = \exp(-\mathrm{i}\alpha n)\Psi_n(x) \tag{5}$$

with eigenvalue $\exp(-\mathrm{i}\alpha n)$. Note that the kernel of the fractional FT equals the propagator of the non-stationary Schrödinger equation for the harmonic oscillator, except for the factor $\exp(\mathrm{i}\alpha/2)$.

It has been proven [3] that if $\alpha/2\pi$ is an irrational number then the Hermite–Gauss functions are the only solutions of (3).

If $\alpha/2\pi$ is rational we can represent an angle α in the form $\alpha = 2\pi N/M$, where N and M are relatively prime integers and $N < M$. Then as was proved in [2], a SFFF for any angle $\alpha = 2\pi N/M$ is also one for the angle $2\pi/M$ and vice versa. This allows us to define a SFFF of order M, which is an eigenfunction of the fractional Fourier transform operator $R^{2\pi/M}$.

From the additive property for fractional FT $R^\alpha R^\beta = R^{\alpha+\beta}$[10] and (3) it immediately follows that if a function is a SFFF for $\alpha = 2\pi/M$ with eigenvalue A it is also one for αk ($k = 1, 2, \ldots$) with eigenvalue A^k. Moreover, taking into account the Parseval relation for the fractional FT [10] and the fact that the fractional FT is periodic: $R^{2\pi}[f(x)](u) = f(u)$ implies that $A^M = 1$, so we get that $A = \exp(\pm\mathrm{i}2\pi L/M)$ where $L = 1, \ldots, M$.

3. Decomposition on the self-fractional Fourier functions

It has been shown in [7] that any Fourier transformable function is the linear combination of four self-Fourier functions. Let us prove that any function from L^2 can be represented as a linear combination of M orthogonal SFFFs of order M.

Because the Hermite–Gauss functions form a complete set in the L^2 space, any function $g(u)$ that belongs to this space can be expanded as

$$g(u) = \sum_{n=0}^{\infty} g_n\Psi_n(u). \tag{6}$$

Subdividing the series into M partial ones we have

$$g(u) = \sum_{L=0}^{M-1}\left(\sum_{m=0}^{\infty} g_{L+mM}\Psi_{L+mM}(u)\right) \stackrel{\mathrm{def}}{=} \sum_{L=0}^{M-1} f(u)_{M,L}. \tag{7}$$

According to (5) the fractional FT of $f(u)_{M,L}$ at the angle $\alpha = 2\pi/M$ is

$$R^{2\pi/M}[f(u)_{M,L}](x) = \sum_{m=0}^{\infty} g_{L+mM}\Psi_{L+mM}(u)\exp\left(\frac{-\mathrm{i}2\pi(L+mM)}{M}\right)$$
$$= \exp\left(\frac{-\mathrm{i}2\pi L}{M}\right) f(x)_{M,L}. \tag{8}$$

This means that

$$f(u)_{M,L} \stackrel{\text{def}}{=} \sum_{m=0}^{\infty} g_{L+mM}\Psi_{L+mM}(u) \tag{9}$$

is a SFFF at angle $2\pi/M$ with eigenvalue $\exp(-\mathrm{i}2\pi L/M)$. Because the SFFF of order M but different L are expanded into disjoint series of Hermite–Gauss functions they are orthogonal to each other

$$\int_{-\infty}^{\infty} f(x)_{M,L} f(x)^*_{M,J}\,\mathrm{d}x = \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} g_{L+mM}g^*_{J+jM}\int_{-\infty}^{\infty}\Psi_{L+mM}(x)\Psi^*_{J+jM}(x)\,\mathrm{d}x$$
$$= |g_{L+mM}|^2\delta_{LJ} \tag{10}$$

where * means the complex conjugate. So we can conclude that any function $g(u)$ from L^2 can be represented as a sum of M orthogonal SFFFs of order M (7). In particular for $M = 2$ the sum (7) is the well known decomposition into the even ($L = 0$) and odd ($L = 1$) functions, for $M = 4$ it is the linear combination of four self-Fourier functions [7]. In the limit case $M \to \infty$ we have the Hermite–Gauss expansion (6). Note that, because of (7) and the property that a SFFF for $\alpha = 2\pi/M$ is also a SFFF for $k\alpha$, the fractional FT of $g(u)$ at angle $2\pi k/M$ is a linear combination of the SFFFs in (9) with some phase shifts

$$R^{2\pi k/M}[g(u)](x) = \sum_{L=0}^{M-1} f(u)_{M,L}\exp\left(\frac{-\mathrm{i}2\pi kL}{M}\right) \tag{11}$$

where $k = 0, 1, \ldots$.

4. Synthesis of self-fractional Fourier functions and mode selection

It has been shown in [2] that a SFFF of order M can be constructed from any generator function $g(u) \in L^2$ through the following procedure

$$f(x)_{M,L} = C\sum_{k=1}^{M}\exp\left(\frac{\mathrm{i}2\pi Lk}{M}\right) R^{2\pi(k-1)/M}[g(u)](x) \tag{12}$$

where C is an arbitrary constant.

Let us now pay attention to the physical meaning of this procedure. Substituting in relation (12) the generator function $g(u)$ in the form (6), and using (5), we have

$$f(x)_{M,L} = C\sum_{n=0}^{\infty} g_n\Psi_n(x)\sum_{k=1}^{M}\exp\left(\frac{\mathrm{i}2\pi}{M}(Lk-(k-1)n)\right)$$
$$= C\sum_{n=0}^{\infty} g_n\Psi_n(x)\exp\left(-\frac{\mathrm{i}2\pi L}{M}\right)\sum_{k=1}^{M}\exp\left(\frac{\mathrm{i}2\pi}{M}(k-1)(L-n)\right). \tag{13}$$

It is easy to see that summarizing over k gives 0 for any n except when

$$n - L = mM \tag{14}$$

where m is an integer. In this case it gives M. So we can write

$$f(x)_{M,L} = CM \exp\left(-\frac{\mathrm{i}2\pi L}{M}\right) \sum_{m=0}^{\infty} g_{L+mM} \Psi_{L+mM}(x). \tag{15}$$

Then choosing for normalization $C = M^{-1} \exp(\mathrm{i}2\pi L/M)$ we have

$$f(x)_{M,L} = \sum_{m=0}^{\infty} g_{L+mM} \Psi_{L+mM}(x). \tag{16}$$

Thus the procedure for the construction of the SFFF means the selection of modes of generator function in according to the rule (14). It is easy to see (as was considered in the previous section) that the SFFF of order M but different L are orthogonal to each other even if they were constructed from different generator functions.

Comparing the relations (6) and (16) one can represent the generator function through the sum of M orthogonal SFFFs of order M

$$g(x) = \sum_{L=0}^{M-1} f(x)_{M,L} \tag{17}$$

where

$$f(x)_{M,L} = \frac{1}{M} \sum_{k=1}^{M} \exp\left(\frac{\mathrm{i}2\pi L(k-1)}{M}\right) R^{2\pi(k-1)/M}[g(u)](x). \tag{18}$$

Let us now consider a limit $M \to \infty$. It means that a function should be a SFFF for any α. It is well known that such functions are the Hermite–Gauss functions (4). In the limit case we can put $\Delta\alpha = 2\pi/M$ and write (18) as

$$\lim_{M\to\infty} f(x)_{M,L} = f(x)_{\infty,L} = \frac{1}{2\pi} \int_0^{2\pi} \exp(\mathrm{i}\alpha L) R^{\alpha}[g(u)](x)\, \mathrm{d}\alpha. \tag{19}$$

Then using the representation (13) and (5) we find that

$$\begin{aligned} f(x)_{\infty,L} &= \frac{1}{2\pi} \int_0^{2\pi} \exp(\mathrm{i}\alpha L) R^{\alpha}\left[\sum_{n=0}^{\infty} g_n \Psi_n(u)\right](x)\, \mathrm{d}\alpha \\ &= \sum_{n=0}^{\infty} g_n \Psi_n(x) \frac{1}{2\pi} \int_0^{2\pi} \exp(\mathrm{i}\alpha(L-n))\, \mathrm{d}\alpha = g_L \Psi_L(x). \end{aligned} \tag{20}$$

Thus the integration of the fractional FT at angle α of any function $g(u)$ with an additional phase shift $\exp(\mathrm{i}\alpha L)$ over one period of the parameter α selects mode L in sum (6)

$$\frac{1}{2\pi} \int_{\alpha_0}^{\alpha_0+2\pi} \exp(\mathrm{i}\alpha L) R^{\alpha}[g(u)](x)\, \mathrm{d}\alpha = g_L \Psi_L(x). \tag{21}$$

g_L is given by

$$g_L = \int_{-\infty}^{\infty} g(u) \Psi_L(u)\, \mathrm{d}u. \tag{22}$$

In particular, for $L = 0$, we have that the integration of the fractional FT of any function $g(u)$ from L^2 on α gives the Gauss function

$$\frac{1}{2\pi} \int_0^{2\pi} R^{\alpha}[g(u)](x)\, \mathrm{d}\alpha = g_0 \exp(-x^2/2) \tag{23}$$

where

$$g_0 = \pi^{-1/4} \int_{-\infty}^{\infty} g(u) \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u.$$

So we can conclude that the synthesis of the SFFF from a generator $g(u)$ has the physical meaning of a mode selection. Thus the SFFF of order M and index L contains only the modes $L + mM$ of the generator function, expanded in Hermite–Gauss functions, where $m = 0, 1, \ldots$. For infinite M, this reduces to a single mode which is the L-Hermite–Gauss function.

We finally note that the previous results can be easily extended to the multidimensional case if the fractional Fourier operator acts independently on the different coordinates.

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